Non-radial solutions of the problem $-\Delta u = |u|^{4/(n-2)}u$ in $\mathbb{R}^n$, $n \geq 3$

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Abstract We prove the existence of an infinite sequence of distinct non-radial nodal $G-$invariant solutions for the following critical nonlinear elliptic problem:

(P) $\quad -\Delta u = |u|^{4/(n-2)}u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3$

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1 Introduction

In this paper the main objective is to prove the existence of non-radial nodal solutions for the following critical nonlinear elliptic problem:

(P) $\quad -\Delta u = |u|^{4/(n-2)}u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3$

In problem (P) solutions were obtained, such that $\int_{\mathbb{R}^n} |\nabla u|^2 dx \to \infty$. However, in order to achieve this, we must solve the following problem of interest:

$$(\mathrm{P}_\varepsilon) \quad \begin{cases} -\Delta u_\varepsilon + \varepsilon A(x)u_\varepsilon = F(x)|u_\varepsilon|^{4/(n-2)}u_\varepsilon, \quad n \geq 3 \\ u_\varepsilon \not\equiv 0 \ \text{ in } \Omega_\varepsilon, \ u_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \end{cases},$$

where Ω_ε is an expanding domain in $\mathbb{R}^n$, $n \geq 3$, invariant under the action of a subgroup G of the isometry group $O(n)$ and A, F are two smooth $G-$invariant functions.

Let $2^* = (2n)/(n-2)$ be the critical exponent for the Sobolev imbedding

$$H_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$$

In problem (P_ε) the main difficulty comes because the exponent

$$2^* = \frac{2n}{n-2} = (\frac{4}{n-2} + 1) + 1$$

is critical.
Since the Sobolev imbedding

$$H_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for any

$$q < 2^* = \frac{2n}{n-2}$$

but if

$$q \geq 2^* = \frac{2n}{n-2}$$

is only continuous, in our case, we have to solve a variational problem with lack of compactness.
By lack of compactness, we mean that the functionals that we consider do not satisfy the Palais-Smale condition, (see [28]) (i.e. there exists a sequence along which the functional remains bounded, its gradient goes to zero, and does not converge). The symmetry property of the domain allows us to improve the Sobolev imbedding in higher L^p spaces and to overcome this obstruction.

Problem (P_ε) has been studied by many authors (i.e., see [3], [4], [5], [8], [9], [10], [12], [13], [16], [18], [21], [22], [30], [31] and the references therein). Some special cases, also have been studied. For example, no solution can exist if Ω is starshaped, as a consequence of the Pohozaev identity (see [33]). Furthermore, if Ω is an annulus there exist infinite solutions (see [27]). Also, a general result of Bahri and Coron guarantees the existence of positive solutions in domains Ω having nontrivial topology (i.e. certain homology

groups of Ω are non trivial) (see [6]). The existence and multiplicity of positive or nodal solutions of critical equations on bounded domains or in some contractible domains have been determined by other authors (i.e., see [4], [15], [17], [18], [21] [29], [32], [34]). Some more nonexistence results in this case are available, (see [1], [4], [11], [23], [24], [30], [31]).

The limit problem (P) presents some extra difficulty because of the lack of compactness in unbounded domains. This obstacle can be overcome by obtaining the solutions of this limit problem as the limits of the solutions of a sequence of the problems (P_ε).
Concerning to the problem (P) it is well known that in 1979, Gidas, Ni and Nirenderg (see [20]) proved that any positive solution of this elliptic problem, which has finite energy, namely

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx < \infty$$

is necessarily of the form

$$u(x) = \left(\frac{\sqrt{n(n-2)\alpha}}{a^2 + |x-\xi|^2} \right)^{(n-2)/2} \qquad (*)$$

where $\alpha > 0, \xi \in \mathbb{R}^n$.
Since the equation

$$-\Delta u = |u|^{4/(n-2)} u, \;\; n \geq 3$$

is invariant under the conformal transformations of $\mathbb{R}^n$, if u(x) is a solution, then for any $\lambda > 0$ and $\xi \in \mathbb{R}^n$,

$$\lambda^{(n-2)/2} u(\frac{x-\xi}{\lambda})$$

is, also, a solution.
Moreover, all solutions obtained in this way have the same energy and we will say that these solutions are equivalent. In particular, the solutions $(*)$ are equivalent.
In 1986, Ding (see [16]) used Ambrosetti and Rabinowitz analysis (see [2]) to prove that this problem has infinite distinct solutions $u \in C^2(\mathbb{R}^n)$, with finite energy and which changes sign, but he did not specify the type of these solutions.

In 2004, Bartsch and Schneider (see [7]) proved that for $N > 2m$ the equation $(-\Delta)^m = |u|^{4m/(N-2m)}u$ on $\mathbb{R}^N$ has a sequence of nodal, finite energy solutions which is unbounded in $\mathcal{D}^{m,2}(\mathbb{R}^N)$. This generalizes the result of Ding for $m = 1$, and provides interesting information concerning the number and the kind of the solutions of the equation (see Remark 3.3).
Recently, Gazzini and Musina in [19] (see Corollary 4.4) proved that, for $N \geq 7$, the problem

$$\begin{cases} -\Delta v = v^{2^*-1}, \text{ in } \mathbb{R}^n \\ \qquad v > 0 \end{cases}$$

has a smooth cylindrically symmetric solution $v_\infty : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{N-3} \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} |\nabla v_\infty|^2 dx = +\infty, \quad \int_{\mathbb{R}^n} |v_\infty|^{2^*} dx < S^{N/2}$$

where S is the Sobolev constant.

In this paper our goal is to specify the kind and the number of solutions of the problem (P). We prove the existence of a sequence $\{u_k\}$ of non-radial, inequivalent, nodal $G-$invariant solutions of (P), such that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 dx \to \infty \quad \text{as} \quad k \to \infty$$

Our proof is via approximation of the problem on symmetric bounded domains. This method is different from that used by the previously referenced authors and can be used to solve polyharmonic equations with supercritical exponent and even in the critical of supercritical case, providing an alternative way of utilizing the best constants of the Sovolev inequalities. Furthermore it enables us to determine the kind and the number of solutions of the problem.

2 Resolution of the problem (P_ε)

Let Ω be a bounded, smooth, domain of $\mathbb{R}^n$, $n \geq 3$, $G-$invariant under the action of a compact subgroup G of the isometry group $O(n)$, without finite subgroup.
Such $\Omega's$ in $\mathbb{R}^n$ are the following examples:

Example 1. Let $\overline{T}$ be the three dimensional solid torus

$$\overline{T} = \left\{ (x, y, z) \in \mathbb{R}^3 \setminus (0,0,0) : \left(\sqrt{x^2 + y^2} - r \right)^2 + z^2 \leq r^2, \ r > 0 \right\}$$

with the metric induced by the $\mathbb{R}^3$ metric. Let $G = O(2) \times I$ be the group of rotations around axis z. Then $\overline{T}$ is a bounded, smooth, domain of $\mathbb{R}^3$, invariant under the action of the subgroup G of the isometry group $O(3)$.

Example 2. Let Ω be a bounded, smooth, domain of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $k \geq 2, \ n - k \geq 1$ such that

$$\overline{\Omega} \subset \left(\mathbb{R}^k \backslash \{0\} \right) \times \mathbb{R}^{n-k}$$

Suppose that $\overline{\Omega}$ is invariant under the action of $G_{k,n-k}$, that is

$$\tau(\overline{\Omega}) = \overline{\Omega}, \quad \text{for all} \ \ \tau \in G_{k,n-k},$$

where $G_{k,n-k} = O(k) \times Id_{n-k}$ is the subgroup of the isometry group $O(n)$ of the type

$$(x_1, x_2) \longrightarrow (\sigma(x_1), x_2), \quad \sigma \in O(k), \quad x_1 \in \mathbb{R}^k, \quad x_2 \in \mathbb{R}^{n-k}$$

Then $\overline{\Omega}$ is a bounded, smooth, domain of $\mathbb{R}^n$, invariant under the action of the subgroup $G_{k,m}$ of the isometry group $O(n)$.

We define the Sobolev space $H_1^2(\Omega)$ as the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_1^2(\Omega)} = \left(\|\nabla u\|_2^2 + \|u\|_2^2 \right)^{1/2},$$

the Sobolev space $\overset{\circ}{H}{}_1^2(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H_1^2(\Omega)$ and we denote by $H_{1,G}^2(\Omega)$ and $\overset{\circ}{H}{}_{1,G}^2(\Omega)$ the subspaces of $H_1^2(\Omega)$ and $\overset{\circ}{H}{}_1^2(\Omega)$ of all $G-$invariant functions, respectively.

Consider the following problem

$$(\mathrm{P}_0) \qquad \begin{cases} -\Delta u + a(x)u = f(x)|u|^{q-2}u \\ u > 0 \ \text{ in } \ \Omega, \ u = 0 \text{ on } \partial\Omega \end{cases},$$

where Ω is defined as above and a, f are two smooth $G-$invariant functions. For any small $\varepsilon > 0$ and some $m > 0$ we consider the family of expanding domains

$$\Omega_\varepsilon = \varepsilon^{-m}\Omega = \{\varepsilon^{-m}x : x \in \Omega\}$$

We consider, also, the transformation

$$\phi : \Omega \to \Omega_\varepsilon \ : \ X = \varepsilon^{-m}x, \ x \in \Omega, \ X \in \Omega_\varepsilon \tag{1}$$

and for $l > 0$, we set

$$u_\varepsilon(X) = \varepsilon^{-l}u\left(\varepsilon^m X\right)$$

In particular we obtain

$$|\nabla u| = \varepsilon^{-m}|\nabla u_\varepsilon| \tag{2}$$

and

$$\Delta u = \varepsilon^{-2m}\ \Delta u_\varepsilon \tag{3}$$

Applying the transformation (1) in the equation of the problem (P_0), because of (2) and (3), we obtain the equation

$$-\Delta u_\varepsilon + \varepsilon^{2m}A(x)u_\varepsilon = \varepsilon^{2m+l(2-q)}F(x)|u_\varepsilon|^{q-2}u_\varepsilon$$

Since l is an arbitrary positive real, we can choose $l = \frac{2m}{q-2}$ and thus we have:

$$-\Delta u_\varepsilon + \varepsilon^{2m}A(x)u_\varepsilon = F(x)|u_\varepsilon|^{q-2}u_\varepsilon \tag{4}$$

From the equation (4), setting $q = 2^* = \frac{2n}{n-2}$ and replacing the ε^{2m} by ε, we obtain:

$$-\Delta u_\varepsilon + \varepsilon A(x)u_\varepsilon = F(x)|u_\varepsilon|^{4/(n-2)}u_\varepsilon$$

So, we have to solve the following critical problem

$$(\mathrm{P}_\varepsilon) \qquad \left\{ \begin{array}{c} -\Delta u_\varepsilon + \varepsilon A(x)u_\varepsilon = F(x)|u_\varepsilon|^{4/(n-2)}u_\varepsilon, \quad n \geq 3 \\ u_\varepsilon \not\equiv 0 \ \text{ in } \ \Omega_\varepsilon, \ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \end{array} \right.$$

Consider the functional

$$J(u_\varepsilon) = \int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^2 + \varepsilon A(x)u_\varepsilon^2)dx$$

and suppose that the operator

$$L(u_\varepsilon) = -\Delta u_\varepsilon + \varepsilon A(x)u_\varepsilon$$

is coercive.
That is, there exists a real number $\lambda > 0$, such that

$$J(u_\varepsilon) \geq \lambda \int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^2 + u_\varepsilon^2) dx$$

for all $u_\varepsilon \in \overset{\circ}{H}{}_1^2(\Omega_\varepsilon)$.
For example, the operator L is coercive if $A(x) \geq 0,\ \forall x \in \Omega_\varepsilon$, and more generally when $A(x)$ is greater than minus the best Poincaré constant of $\overset{\circ}{H}{}_1^2(\Omega_\varepsilon)$.

Denote

$$\mathcal{H} = \left\{ u_\varepsilon \in \overset{\circ}{H}{}_{1,G}^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} F(x) u_\varepsilon^2 dx = 1 \right\}.$$

and suppose that exists an isometry σ such that $\sigma(\Omega_\varepsilon) = \Omega_\varepsilon$. Moreover, we suppose that the functions $A(x)$ and $F(x)$ are invariant under the action of σ, and

$$\mathcal{H}_\sigma = \mathcal{H} \cap \left\{ u_\varepsilon \in \overset{\circ}{H}{}_1^2(\Omega_\varepsilon) : u_\varepsilon \circ \sigma = -u_\varepsilon \right\} \neq \varnothing$$

By definition, a function u which verify $u \circ \sigma = -u$ is called antisymmetrical.

Under the above considerations the following theorem holds (see [14]) :

Theorem 2.1 *The problem* (P_ε)*, always, has a non-radial nodal* $\mathcal{H}_\sigma-$*invariant solution. Moreover, if* $F(x) > 0, \quad \forall x \in \overline{\Omega}_\varepsilon$, (P_ε) *has an infinity sequence* $\{u_{\varepsilon_i}\}$ *of non-radial nodal* $\mathcal{H}_\sigma-$*invariant solutions, such that*

$$\lim_{i \to \infty} \int_{\Omega_\varepsilon} (|\nabla u_{\varepsilon_i}|^2 + u_{\varepsilon_i}^2) dx = \infty$$

Remark 2.1 Following the same arguments, we can prove that the above Theorem 2.1 holds in the supercritical case, where

$$\frac{2n}{n-2} < q < \frac{2(n-k+1)}{n-k-1}$$

3 Resolution of the problem (P)

Because of the double lack of compactness, direct variational methods are not applicable to the limit problem

$$\text{(P)} \qquad -\Delta u = |u|^{4/(n-2)}u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3$$

However, this method is successful in approximating a solution to the problem (P) by solutions considered in the open domains Ω_{ε_j}. Thus a solution to (P) may be then obtained by the limit procedure as $\varepsilon_j \to 0$.

Before we will approximate the solutions in $\mathbb{R}^n$ by solutions in bounded domains $\Omega_\varepsilon \in \mathbb{R}^n$, we note that, in the generalized setting of the problems in Ω's, the Dirichlet condition $u_\varepsilon(x) = 0$ on $\partial\Omega_\varepsilon$ may actually included in the condition $u_\varepsilon \in \overset{\circ}{H}{}_1^2(\Omega_\varepsilon)$.

Moreover, since any function $u_\varepsilon \in \overset{\circ}{H}{}_1^2(\Omega_\varepsilon)$ can be extended onto $\mathbb{R}^n$ by

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in \Omega_\varepsilon \\ 0, & x \in \mathbb{R}^n \backslash \Omega_\varepsilon \end{cases},$$

generalized solutions may be defined in Ω_ε's analogously to the case in $\mathbb{R}^n$ in the following way:

Definition 3.1 *A function $u_\varepsilon \in \overset{\circ}{H}{}_1^2(\Omega_\varepsilon)$ is a generalized solution of* (P_ε) *if the function $f(x, u_\varepsilon(x)) = \varepsilon A(x)u_\varepsilon - F(x)|u_\varepsilon|^{4/(n-2)}u_\varepsilon$ is locally integrable and for all $\varphi \in C_0^\infty(\Omega_\varepsilon)$,*

$$\int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla\varphi)dx + \int_{\Omega_\varepsilon} f(x, u_\varepsilon)\varphi dx = 0$$

Definition 3.2 *A function $u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C(\overline{\Omega}_\varepsilon)$ is a classical solution to* (P_ε) *if after substituting it into equation of* (P_ε)*, this equation becomes the identity at each $x \in \Omega_\varepsilon$ and $u_\varepsilon(x) = 0$ provided $x \in \partial\Omega_\varepsilon$.*

Consider now a sequence of real numbers $\{\varepsilon_j\}_{j=1,2,..}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and the sequence of problems

$$\left(\mathrm{P}_{\varepsilon_j}\right) \qquad \begin{cases} -\Delta u_{\varepsilon_j} + \varepsilon A(x)u_{\varepsilon_j} = F(x)|u_{\varepsilon_j}|^{4/(n-2)}u_{\varepsilon_j}, \quad n \geq 3 \\ \qquad u_{\varepsilon_j} \not\equiv 0 \text{ in } \Omega_{\varepsilon_j}, \ u_{\varepsilon_j} = 0 \text{ on } \partial\Omega_{\varepsilon_j} \end{cases}$$

where $F(x) > 0, \quad \forall x \in \overline{\Omega}_{\varepsilon_j}$.

According to the above Theorem 2.1, every problem $(\mathrm{P}_{\varepsilon_j})$ has an infinity sequence of nodal $\mathcal{H}_\sigma-$invariant solutions $\{(u_{\varepsilon_j})_{k,k=1,2,\ldots}\}$, such that

$$\lim_{k\to\infty} \int_{\Omega_\varepsilon} (|\nabla u_{\varepsilon_{j_k}}|^2 + u_{\varepsilon_{j_k}}^2)dx = \infty \tag{5}$$

Let $\{u_j\}$ an arbitrary sequence of such solutions, such that $u_j \in (u_{\varepsilon_j})_{k,\, k=1,2,..}$ for all $j \in \mathbb{N}^*$. Then the following theorem on approximation by bounded domains holds:

Theorem 3.1 *The problem*

$$-\Delta u = F(x)|u|^{4/(n-2)}u \quad \text{in } \mathbb{R}^n, \quad n \geq 3$$

has a generalized non-radial nodal $\mathcal{H}_\sigma-$invariant solution u and there is a subsequence $\{u_j\}$, such that

$$u_j \rightharpoonup u \quad \text{in} \quad H^2_{1,G} \quad \text{as} \quad j \to \infty$$

The proof of the Theorem 3.1 arises immediately by the Vainberg-Krasnoselskii Theorem (see [35] or [25]) and by the Theorem 3.2 in [26].

Corollary 3.1 *The problem*

$$\text{(P)} \qquad -\Delta u = |u|^{4/(n-2)}u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3$$

has a sequence $\{u_k\}$ of non-radial nodal $\mathcal{H}_\sigma-$invariant solutions, such that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 dx \to \infty \quad \text{as} \quad k \to \infty$$

Remark 3.1 The number of the sequences of non-radial nodal $\mathcal{H}_\sigma-$invariant solutions to the problem (P) depends on the number of all subgroups of $O(n)$ of which the cardinal of orbits with minimum volume is infinite, that is on the dimension n of the domain.

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